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Group properties and new solutions of Navier-Stokes equations

R.E. BOISVERT *,**

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, U.S.A.

W.F. AMES *,***

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, U.S.A.

and

U.N. SRIVASTAVA ***

Department of Mathematics, India Institute of Technology, Madras-36, India

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Summary

Using the machinery of Lie theory (groups and algebras) applied to the Navier-Stokes equations a number of exact solutions for the steady state are derived in (two) three dimensions. It is then shown how each of these generates an infinite number of time-dependent solutions via (three) four arbitrary functions of time. This algebraic structure also provides the mechanism to search for other solutions since its character is inferred from the basic equations.

1. Introduction

The group-theoretic methods introduced by Lie [1,2], as amplified by many authors (see Ovsiannikov [3] and Ames [4]) are applied to the Navier-Stokes equations in Cartesian coordinates

$$u_t + uu_x + vu_y + wu_z = -p_x + \mu\nabla^2 u, \tag{1.1}$$

$$v_t + uv_x + vv_y + wv_z = -p_y + \mu\nabla^2 v, \tag{1.2}$$

$$w_t + uw_x + vw_y + ww_z = -p_z + \mu\nabla^2 w, \tag{1.3}$$

where u, v, w are the three velocity components, p is pressure and μ is (constant) viscosity.

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** Present address: M.I.T. Lincoln Laboratory, Lexington, MA 02173, U.S.A.

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To these must be added the continuity equation

$$u_x + v_y + w_z = 0. \tag{1.4}$$

The unsteady state is analyzed to obtain the full transformation group admitted by the equations. In turn these groups are utilized to obtain several new exact solutions of the equations in both two and three dimensions.

Development of the full transformation group for the two-dimensional equations was first examined by Puhnachev [5]. In three dimensions, Bitev [6], and later Lloyd [7], give the first derivation. The work of these authors is verified and it is shown how the group permits the association of an infinite number of time-dependent solutions to any steady-state solution.

In Section 2 the full Lie group and Lie algebra are discussed. The complete derivation can be found in Bitev [6], Lloyd [7], and Boisvert [8]. In Sections 3 and 4 the two-dimensional solutions will be discussed followed by the three-dimensional case in Sections 5 and 6. These solutions are generally singular in nature. This paper expands on the results presented by the authors in [9].

2. Full Lie group and algebra

In the spirit of Lie it is desired to find infinitesimal transformations of the form

$$\begin{aligned}
t' &= t + \varepsilon T(t, x, y, z, u, v, w, p) + O(\varepsilon^2),\\
x' &= x + \varepsilon X(t, x, y, z, u, v, w, p) + O(\varepsilon^2),\\
y' &= y + \varepsilon Y(t, x, y, z, u, v, w, p) + O(\varepsilon^2),\\
z' &= z + \varepsilon Z(t, x, y, z, u, v, w, p) + O(\varepsilon^2),\\
u' &= u + \varepsilon U(t, x, y, z, u, v, w, p) + O(\varepsilon^2),\\
v' &= v + \varepsilon V(t, x, y, z, u, v, w, p) + O(\varepsilon^2),\\
w' &= w + \varepsilon W(t, x, y, z, u, v, w, p) + O(\varepsilon^2),\\
p' &= p + \varepsilon P(t, x, y, z, u, v, w, p) + O(\varepsilon^2),
\end{aligned} \tag{2.1}$$

which leave (1.1–1.4) invariant. System (2.1) leaves (1.1–1.4) invariant if and only if (u', v', w', p') is a solution of (1.1′–1.4′) whenever (u, v, w, p) is a solution to (1.1–1.4). By (1.1′–1.4′) is meant the same equations in the primed variables. By extensive analysis the following theorem is established (see Boisvert [8]):

Theorem. The full Lie group which leaves the Navier-Stokes equations (1.1–1.4) invariant is given by (2.1) with

$$T = \alpha + 2\beta t, \tag{2.2}$$

$$X = \beta x - \gamma y - \lambda z + f(t), \tag{2.3}$$

$$Y = \beta y + \gamma x - \sigma z + g(t), \tag{2.4}$$

$$Z = \beta z + \lambda x + \sigma y + h(t), \tag{2.5}$$

$$U = -\beta u - \gamma v - \lambda w + f'(t), \tag{2.6}$$

$$V = -\beta v + \gamma u - \sigma w + g'(t), \tag{2.7}$$

$$W = -\beta w + \lambda u + \sigma v + h'(t), \tag{2.8}$$

$$P = -2\beta p + j(t) - xf''(t) - yg''(t) - zh''(t), \tag{2.9}$$

where α, β, γ, λ and σ are five arbitrary parameters and $f(t)$, $g(t)$, $h(t)$, and $j(t)$ are arbitrary, sufficiently smooth, functions of t.

Each of the arbitrary parameters of the preceding paragraph corresponds to a well-known transformation. The parameter α corresponds to a translation in time; β represents a stretching (dilatation) in all coordinates; γ, λ, σ represent rotations of the spatial system. With $f(t)$, $g(t)$, $h(t)$ as constants it is clear that translations in the various coordinate directions are also included. Moving coordinate transformations are also included as long as these changes are reflected in U, V, W, and P, as shown in (2.6), (2.7), (2.8) and (2.9). The classical Galilei-Newton group is obviously a subgroup of this full group.

The infinitesimal operator (generator of the Lie algebra) associated with each parameter is obtained from the operator

$$Q = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + W\frac{\partial}{\partial w} + P\frac{\partial}{\partial p} \tag{2.10}$$

by setting the studied parameter equal to one while all other parameters and arbitrary functions are equated to zero. The operator associated with each of the arbitrary functions in (2.2) to (2.9) is obtained by setting the other arbitrary functions and all parameters identically equal to zero.

With X_i, $i = 1, 2, 3, 4, 5$ representing the generators associated with the parameters α, β, γ, λ, and σ, respectively, it follows that

$$X_1 = \frac{\partial}{\partial t}, \tag{2.11}$$

$$X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p}, \tag{2.12}$$

$$X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \tag{2.13}$$

$$X_4 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} - w\frac{\partial}{\partial u} + u\frac{\partial}{\partial w}, \tag{2.14}$$

$$X_5 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} - w\frac{\partial}{\partial v} + v\frac{\partial}{\partial w}. \tag{2.15}$$

Table 1.
Commutator table of the Lie algebra for the Navier-Stokes equations

	X_1	X_2	X_3	X_4	X_5	$X_6(f_1)$	$X_7(g_1)$	$X_8(h_1)$	$X_9(j_1)$
X_1	0	$2X_1$	0	0	0	$X_6(f_1')$	$X_7(g_1')$	$X_8(h_1')$	$X_9(j_1')$
X_2	$-2X_1$	0	0	0	0	$X_6(2tf_1'-f_1)$	$X_7(2tg_1'-g_1)$	$X_8(2th_1'-h_1)$	$X_9(2tj_1'+2j_1)$
X_3	0	0	0	$-X_5$	X_4	$-X_7(f_1)$	$X_6(g_1)$	0	0
X_4	0	0	X_5	0	$-X_3$	$-X_8(f_1)$	0	$X_6(h_1)$	0
X_5	0	0	$-X_4$	X_3	0	0	$-X_8(g_1)$	$X_7(h_1)$	0
$X_6(f_2)$	$-X_6(f_2')$	$-X_6(2tf_2'-f_2)$	$X_7(f_2)$	$X_8(f_2)$	0	0	0	0	0
$X_7(g_2)$	$-X_7(g_2')$	$-X_7(2tg_2'-g_2)$	$-X_6(g_2)$	0	$X_8(g_2)$	0	0	0	0
$X_8(h_2)$	$-X_8(h_2')$	$-X_8(2th_2'-h_2)$	0	$-X_6(h_2)$	$-X_7(h_2)$	0	0	0	0
$X_9(j_2)$	$-X_9(j_2')$	$-X_9(2tj_2'+2j_2)$	0	0	0	0	0	0	0

Also, from the arbitrary functions in (2.2)–(2.9), there are obtained infinitely many operators of the following forms:

$$X_6(f)=f(t)\frac{\partial}{\partial x}+f'(t)\frac{\partial}{\partial u}-xf''(t)\frac{\partial}{\partial p}, \tag{2.16}$$

$$X_7(g)=g(t)\frac{\partial}{\partial y}+g'(t)\frac{\partial}{\partial v}-yg''(t)\frac{\partial}{\partial p}, \tag{2.17}$$

$$X_8(h)=h(t)\frac{\partial}{\partial z}+h'(t)\frac{\partial}{\partial w}-zh''(t)\frac{\partial}{\partial p}, \tag{2.18}$$

$$X_9(j)=j(t)\frac{\partial}{\partial p}. \tag{2.19}$$

The operators (2.11)–(2.15) generate a finite-dimensional Lie algebra L_5 which is a five-dimensional subalgebra of the infinite-dimensional algebra L_∞ generated by the operators (2.11)–(2.19). The commutator table of the Lie algebra for the Navier-Stokes equations is given in Table 1, where the entry in the ith row and jth column is the commutator of X_i, X_j, that is

$$[X_i, X_j]=X_iX_j-X_jX_i.$$

3. Two-dimensional solutions

Some exact solutions to the two-dimensional Navier-Stokes equations

$$u_t+uu_x+vu_y=-p_x+\mu\nabla^2u, \tag{3.1}$$

$$v_t+uv_x+vv_y=-p_y+\mu\nabla^2v, \tag{3.2}$$

$$u_x+v_y=0, \tag{3.3}$$

are obtained here by utilizing different subgroups of the full group of the theorem. Associated with these are their invariants obtained by integrating the associated $QI=0$. This permits a reduction by one in the number of independent variables and a possible association with dependent variables.

The reduced full group for (3.1)–(3.3) is

$$\begin{aligned} &T=\alpha+2\beta t, \qquad X=\beta x-\gamma y+f(t), \qquad Y=\beta y+\gamma x+g(t),\\ &U=-\beta u-\gamma v+f'(t), \qquad V=-\beta v+\gamma u+g'(t),\\ &P=-2\beta p+j(t)-xf''(t)-yg''(t). \end{aligned} \tag{3.4}$$

First, consider the subgroup of (3.4) with $\beta=\gamma=0$ and $\alpha=1$. This subgroup, $T=1$, $X=f(t)$, $Y=g(t)$, $U=f'(t)$, $V=g'(t)$, $P=j(t)-xf''(t)-yg''(t)$, has the associated operator

$$Q=\frac{\partial}{\partial t}+f(t)\frac{\partial}{\partial x}+g(t)\frac{\partial}{\partial y}+f'(t)\frac{\partial}{\partial u}+g'(t)\frac{\partial}{\partial v}+\left[j(t)-xf''(t)-yg''(t)\right]\frac{\partial}{\partial p}.$$

The first-order equation for the invariants $QI=0$ has the characteristics

$$\tilde{x}=x-F(t),\qquad \tilde{y}=y-G(t), \tag{3.5}$$

where $F=\int f\mathrm{d}t$, $G=\int g\mathrm{d}t$, while

$$\begin{aligned} &u=\tilde{u}(\tilde{x},\tilde{y})+f(t),\qquad v=\tilde{v}(\tilde{x},\tilde{y})+g(t),\\ &p=\tilde{p}(\tilde{x},\tilde{y})-xf'(t)-yg'(t)+k(t), \end{aligned} \tag{3.6}$$

where

$$k(t)=\tfrac{1}{2}f^2+\tfrac{1}{2}g^2+\int j(t)\mathrm{d}t.$$

The functions $\tilde{u}$, $\tilde{v}$ and $\tilde{p}$ satisfy the steady Navier-Stokes equations

$$\begin{aligned} &\tilde{u}\tilde{u}_{\tilde{x}}+\tilde{v}\tilde{u}_{\tilde{y}}=-\tilde{p}_{\tilde{x}}+\mu\left[\tilde{u}_{\tilde{x}\tilde{x}}+\tilde{u}_{\tilde{y}\tilde{y}}\right],\\ &\tilde{u}\tilde{v}_{\tilde{x}}+\tilde{v}\tilde{v}_{\tilde{y}}=-\tilde{p}_{\tilde{y}}+\mu\left[\tilde{v}_{\tilde{x}\tilde{x}}+\tilde{v}_{\tilde{y}\tilde{y}}\right],\\ &\tilde{u}_{\tilde{x}}+\tilde{v}_{\tilde{y}}=0. \end{aligned} \tag{3.7}$$

Consequently, there is obtained the useful result that any steady-state solution to the two-dimensional equations can be transformed by means of (3.5) and (3.6) into a time-dependent solution involving three arbitrary functions of the time variable. A similar result holds in three dimensions. Another interesting consequence of the transformation is that different subgroups of the reduced (time-independent) full group may now be used to study (3.7) and transform it into a system of ordinary differential equations.

From (3.4) it follows that the dilatation subgroup generated by β (the source of what is often called "similarity variables") will leave (3.7) invariant. The characteristic equations of $QI=0$ are

$$\frac{\mathrm{d}\tilde{x}}{\tilde{x}}=\frac{\mathrm{d}\tilde{y}}{\tilde{y}}=\frac{\mathrm{d}\tilde{u}}{-\tilde{u}}=\frac{\mathrm{d}\tilde{v}}{-\tilde{v}}=\frac{\mathrm{d}\tilde{p}}{-2\tilde{p}},$$

whose invariants are

$$\eta=\tilde{x}/\tilde{y},\qquad \tilde{u}=F_1(\eta)/\tilde{x},\qquad \tilde{v}=G_1(\eta)/\tilde{x},\qquad p=J_1(\eta)/\tilde{x}^2.$$

In these variables equations (3.7) become the ordinary differential equations (primes denote differentiation with respect to η)

$$-F_2^2+\eta F_2F_2'-\eta^2G_2F_2'-2J_2+\eta J_2'-2F_2+2\eta F_2'-\eta^2F_2''-2\eta^3F_2'-\eta^4F_2''=0, \tag{3.8a}$$

$$-F_2G_2 + \eta F_2G_2' - \eta^2 G_2G_2' - \eta^2 J_2' - 2G_2 + 4\eta G_2' - \eta^2 G_2'' - 2\eta^3 G_2' - \eta^4 G_2'' = 0, \quad (3.8b)$$

$$-F_2 + \eta F_2' - \eta^2 G_2' = 0, \quad (3.8c)$$

where $F_2 = \mu^{-1}F_1$, $G_2 = \mu^{-1}G_1$, and $J_2 = \mu^{-1}J_1$.

The last of these equations is satisfied when

$$F_2 = \eta G_2 + c_1\eta. \quad (3.9)$$

Multiplying (3.8b) by η, subtracting (3.8a) from it, and utilizing (3.9) gives

$$-2J_2 + \eta J_2' + \eta^3 J_2' = 2c_1\eta^3 - 2\eta G_2 + 2\eta^3 G_2 + 2\eta^2 G_2' + 2\eta^4 G_2'. \quad (3.10)$$

Setting both sides of (3.10) equal to $\varphi(\eta)$ and then integrating gives

$$J_2 = \frac{\eta^2}{\eta^2+1}\int \frac{\varphi(\eta)}{\eta^3}\,\mathrm{d}\eta$$

and

$$G_2 = \frac{\eta}{2(\eta^2+1)}\int\left[\frac{\varphi(\eta)}{\eta^3} - 2c_1\right]\mathrm{d}\eta.$$

Therefore,

$$J_2 = 2\eta G_2 + \frac{2c_1\eta^3 + c_2\eta^2}{\eta^2+1}$$

will satisfy (3.10). Replacing J_2 and F_2 in (3.8b) yields the second-order ordinary differential equation

$$G_3'' + \frac{G_3^2}{(\eta^2+1)^{5/2}} + \frac{3G_3}{(\eta^2+1)^2} = \frac{c_3\eta^3 + 3c_3\eta + c_4}{(\eta^2+1)^{3/2}}, \quad (3.11)$$

where $G_3 = \{(\eta^2+1)^{3/2}/\eta\}G_2$ and c_3, c_4 are new arbitrary constants.

Any solution of (3.11) will lead back to a solution of the Navier-Stokes equations. Two solutions have been obtained: $G_3 = -6(\eta^2+1)^{3/2}$ for the case $c_3 = c_4 = 0$, and $G_3 = c(\eta^2+1)^{1/2}$ for the case $c_3 = 0$ and $c_4 = -\frac{1}{2}(c^2 + 4c)$. The solutions of system (3.1)–(3.3) corresponding to these are, respectively,

$$u = f(t) - 6\mu(X - F(t))(y - G(t))^{-2},$$

$$v = g(t) - 6\mu(y - G(t))^{-1}, \quad (3.12)$$

$$p = -xf'(t) - yg'(t) + k(t) - 12\mu^2(y - G(t))^{-2},$$

and

$$u = f(t) + c(x - F(t))\left[(x - F(t))^2 + (y - G(t))^2\right]^{-1},$$

$$v = g(t) + c(y - G(t))\left[(x - F(t))^2 + (y - G(t))^2\right]^{-1},$$

$$p = -f'(t)x - g'(t)y + k(t) - \tfrac{1}{2}c^2\left[(x - F(t))^2 + (y - G(t))^2\right]^{-1}.$$

The second solution is independent of the viscosity of the fluid. Solution (3.12) has also been found by Berker [10] using other methods.

An alternative approach is to use the rotation group, generated by the parameter γ in (3.4), in the stream-function form

$$\psi_{\tilde{x}}\psi_{\tilde{y}\tilde{y}\tilde{y}} + \psi_{\tilde{x}}\psi_{\tilde{x}\tilde{x}\tilde{y}} - \psi_{\tilde{y}}\psi_{\tilde{x}\tilde{y}\tilde{y}} - \psi_{\tilde{y}}\psi_{\tilde{x}\tilde{x}\tilde{x}} - \mu\left(\psi_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + 2\psi_{\tilde{x}\tilde{x}\tilde{y}\tilde{y}} + \psi_{\tilde{y}\tilde{y}\tilde{y}\tilde{y}}\right) = 0 \qquad (3.13)$$

of equations (3.7). Here, $\psi_{\tilde{x}} = \tilde{v}$ and $\psi_{\tilde{y}} = -\tilde{u}$. In this setting the rotation group is

$$\tilde{x}' = \tilde{x} - \varepsilon\tilde{y} + O(\varepsilon^2),$$

$$\tilde{y}' = \tilde{y} + \varepsilon\tilde{x} + O(\varepsilon^2),$$

$$\psi' = \psi + O(\varepsilon^2)$$

with invariants,

$$\eta = \tilde{x}^2 + \tilde{y}^2, \qquad \psi = f(\eta).$$

In terms of f and η, equation (3.13) becomes the linear equation

$$2f'' + 4\eta f''' + \eta^2 f'''' = 0 \qquad (3.14)$$

which is independent of μ. The general solution of (3.14) is

$$f(\eta) = -c_1 \ln\eta + c_2\eta\ln\eta + c_3\eta + c_4,$$

where the c_i are arbitrary constants. This solution was first obtained by Hamel [11] and later by others using ad-hoc methods.

The associated time-dependent solution is

$$u = f(t) + (y - G(t))\left[c_1\eta^{-1} - c_5\ln\eta - c_6\right],$$

$$v = g(t) + (x - F(t))\left[-c_1\eta^{-1} + c_5\ln\eta + c_6\right],$$

$$p = -xf'(t) - yg'(t) + k(t) - \tfrac{1}{2}c_1^2\eta^{-1} + \left(c_5^2 - c_5c_6 + \tfrac{1}{2}c_6^2\right)\eta - c_1c_6\ln\eta$$

$$+ \left(c_5c_6 - c_5^2\right)\eta\ln\eta - \tfrac{1}{2}c_1c_5(\ln\eta)^2 + \tfrac{1}{2}c_5^2\eta(\ln\eta)^2 - 4\mu c_5\tan^{-1}\left[\frac{x - F(t)}{y - G(t)}\right],$$

where $\eta = [x - F(t)]^2 + [y - G(t)]^2$.

4. Another set of two-dimensional solutions

Another set of solutions may be obtained by applying the subgroup of (3.4) generated by β, the dilatation subgroup, directly to the unsteady equations (3.1)–(3.3). The invariants associated with this subgroup are

$$\eta_1 = xt^{-1/2}, \qquad \eta_2 = yt^{-1/2}$$

and

$$u = t^{-1/2} f(\eta_1, \eta_2), \qquad v = t^{-1/2} g(\eta_1, \eta_2), \qquad p = t^{-1} h(\eta_1, \eta_2).$$

With these substitutions system (3.1)–(3.3) becomes

$$\begin{aligned} &-\tfrac{1}{2}f - \tfrac{1}{2}\eta_1 f_{\eta_1} - \tfrac{1}{2}\eta_2 f_{\eta_2} + f f_{\eta_1} + g f_{\eta_2} + h_{\eta_1} - \mu\left(f_{\eta_1\eta_1} + f_{\eta_2\eta_2}\right) = 0, \\ &-\tfrac{1}{2}g - \tfrac{1}{2}\eta_1 g_{\eta_1} - \tfrac{1}{2}\eta_2 g_{\eta_2} + f g_{\eta_1} + g g_{\eta_2} + h_{\eta_2} - \mu\left(g_{\eta_1\eta_1} + g_{\eta_2\eta_2}\right) = 0, \\ &f_{\eta_1} + g_{\eta_2} = 0. \end{aligned} \tag{4.1}$$

The Lie group leaving (4.1) invariant is

$$\begin{aligned} &\eta_1' = \eta_1 + \varepsilon c_1 + O(\varepsilon^2), \\ &\eta_2' = \eta_2 + \varepsilon c_2 + O(\varepsilon^2), \\ &f' = f + \varepsilon(c_1/2) + O(\varepsilon^2), \\ &g' = g + \varepsilon(c_2/2) + O(\varepsilon^2), \\ &h' = h + \varepsilon\tfrac{1}{4}(c_1\eta_1 + c_2\eta_2 + c_3) + O(\varepsilon^2), \end{aligned} \tag{4.2}$$

where c_1, c_2, and c_3 are arbitrary constants. The invariants of this Lie group,

$$\eta = c_2\eta_1 - c_1\eta_2, \qquad f = \frac{\eta_1}{2} + F(\eta), \qquad g = \frac{\eta_2}{2} + G(\eta),$$

$$h = \frac{\eta_1^2}{8} - \frac{c_2^2\eta_1^2}{8c_1^2} + \frac{c_2}{4c_1}\eta_1\eta_2 + \frac{c_3\eta_1}{4c_1} + H(\eta),$$

transform (4.1) into the following system of ordinary differential equations:

$$\begin{aligned} &c_2 FF' - c_1 GF' - \frac{c_2}{4c_1^2}\eta + \frac{c_3}{4c_1} + c_2 H' - \mu\left(c_1^2 + c_2^2\right)F'' = 0, \\ &c_2 FG' - c_1 GG' + \frac{1}{4c_1}\eta - c_1 H' - \mu\left(c_1^2 + c_2^2\right)G'' = 0, \\ &c_2 F' - c_1 G' + 1 = 0. \end{aligned} \tag{4.3}$$

A solution to (4.3), for the case $c_3 = 0$, is

$$F = \frac{c_1}{c_2} G - \frac{n}{c_2} + c_4,$$

$$G = \frac{c_1 \eta}{c_1^2 + c_2^2} + c_5 \operatorname{erf}\left[\frac{\eta - c_2 c_4}{\left(2\mu\left(c_1^2 + c_2^2\right)\right)^{1/2}}\right] + c_6,$$

$$H = \left[\frac{c_2^2 - 3c_1^2}{8c_1^2\left(c_1^2 + c_2^2\right)}\right]\eta^2 + \frac{c_2 c_4}{c_1^2 + c_2^2}\eta + c_7.$$

The corresponding solution of the unsteady equations (3.1)–(3.3) is

$$u = \left[\frac{c_1^2 - c_2^2}{2\left(c_1^2 + c_2^2\right)}\right] x t^{-1} + \left[\frac{c_1 c_2}{c_1^2 + c_2^2}\right] y t^{-1} + \left[\frac{c_1 c_6}{c_2} + c_4\right] t^{-1/2}$$

$$+ \frac{c_1 c_5}{c_2}\left[\operatorname{erf}\left[\frac{c_2 x t^{-1/2} - c_1 y t^{-1/2} - c_2 c_4}{\left(2\mu\left(c_1^2 + c_2^2\right)\right)^{1/2}}\right]\right] t^{-1/2},$$

$$v = \left[\frac{c_1 c_2}{c_1^2 + c_2^2}\right] x t^{-1} + \left[\frac{c_2^2 - c_1^2}{2\left(c_1^2 + c_2^2\right)}\right] y t^{-1} + c_6 t^{-1/2}$$

$$+ c_5\left[\operatorname{erf}\left[\frac{c_2 x t^{-1/2} - c_1 y t^{-1/2} - c_2 c_4}{\left(2\mu\left(c_1^2 + c_2^2\right)\right)^{1/2}}\right]\right] t^{-1/2},$$

$$p = \left[\frac{c_1^2 - 3c_2^2}{8\left(c_1^2 + c_2^2\right)}\right] x^2 t^{-2} + \left[\frac{c_2^2 - 3c_1^2}{8\left(c_1^2 + c_2^2\right)}\right] y t^{-2} + \left[\frac{c_1 c_2}{c_1^2 + c_2^2}\right] x y t^{-2} + \left[\frac{c_2^2 c_4}{c_1^2 + c_2^2}\right] x t^{-3/2}$$

$$- \left[\frac{c_1 c_2 c_4}{c_1^2 + c_2^2}\right] y t^{-3/2} + c_7 t^{-1}.$$

The substitutions

$$\eta = \eta_1 - \eta_2, \qquad F(\eta) = f(\eta_1, \eta_2) - \eta_2,$$

$$G(\eta) = g(\eta_1, \eta_2) - \eta_1, \qquad H(\eta) = h(\eta_1, \eta_2),$$

will also reduce (4.1) to a system of ordinary differential equations, though they are not part of the Lie group in (4.2). In these variables (4.1) becomes

$$\eta - \tfrac{3}{2}\eta F' - \tfrac{1}{2}F + FF' + G - GF' + H' - 2\mu F'' = 0, \tag{4.4a}$$

$$-\eta - \tfrac{3}{2}\eta G' - \tfrac{1}{2}G + FG' + F - GG' - H' - 2\mu G'' = 0, \tag{4.4b}$$

$$F' - G' = 0. \tag{4.4c}$$

The last of these equations is satisfied by

$$F = G + c_1. \tag{4.5}$$

Replacing F in (4.4a) and (4.4b) then subtracting the resulting equations yields

$$2H' + 2\eta - \tfrac{3}{2}c_1 = 0,$$

whose solution is

$$H = -\tfrac{1}{2}\eta^2 + \tfrac{3}{4}c_1\eta + c_2.$$

The following second-order equation for G may be obtained by adding (4.4a) to (4.4b) and using (4.5):

$$-4\mu G'' + (2c_1 - 3\eta)G' + G + \tfrac{1}{2}c_1 = 0. \tag{4.6}$$

When $c_1 = 0$, the substitutions

$$\zeta = \tfrac{1}{2}\mu^{-1/2}\eta, \qquad J = e^{(3/4)\zeta^2}G$$

transform (4.6) into

$$\frac{d^2J}{d\zeta^2} + \left(-\tfrac{9}{4}\zeta - \tfrac{5}{2}\right)J = 0, \tag{4.7}$$

which is a special form of Weber's equation (see Moon and Spencer [12, p. 182]). The general solution of (4.7) is given in [12] and is

$$J = c_3 W_e(p, q\zeta) + c_4 W_0(p, q\zeta),$$

where $p = -4/3$, $q = \sqrt{3}$,

$$W_e(p, q\zeta) = e^{-1/4(q\zeta)^2}\left\{1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} p(p-2)(p-4)\dots(p-2[k-1])(q\zeta)^{2k}\right\},$$

$$W_0(p, q\zeta)$$
$$= e^{-1/4(q\zeta)^2}(q\zeta)\left\{1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}(p-1)(p-3)\dots(p-[2k-1])(q\zeta)^{2k}\right\}.$$

Note that W_e is an even function, W_0 is an odd function, and that both series are absolutely convergent.

Following the substitutions back to u, v, and p leads to the solution

$$u = t^{-1/2} e^{-3(x-y)^2/16\mu t}\left[c_3 W_e\left[-\tfrac{4}{3}, \frac{\sqrt{3}(x-y)}{2\sqrt{\mu t}}\right] + c_4 W_0\left[-\tfrac{4}{3}, \frac{\sqrt{3}(x-y)}{2\sqrt{\mu t}}\right]\right] + yt^{-1},$$

$$v=t^{-1/2}\,e^{-3(x-y)^2/16\mu t}\left[c_3W_e\left[-\tfrac{4}{3},\frac{\sqrt{3}(x-y)}{2\sqrt{\mu t}}\right]+c_4W_0\left[-\tfrac{4}{3},\frac{\sqrt{3}(x-y)}{2\sqrt{\mu t}}\right]\right]+xt^{-1},$$

$$p=c_2t^{-1}+\tfrac{1}{2}(x-y)^2t^{-2}.$$

5. Three-dimensional solutions

Different subgroups of the full group given in the theorem will be utilized to construct solutions of the three-dimensional equations. In a manner similar to that of Section 3 (3.5 to 3.7) the time-dependent three-dimensional equations are transformed to the steady-state equations in the variables

$$\tilde{x}=x-F(t),\qquad \tilde{y}=y-G(t),\qquad \tilde{z}=z-H(t)$$

with

$$u=\tilde{u}(\tilde{x},\tilde{y},\tilde{z})+f(t),\qquad v=\tilde{v}(\tilde{x},\tilde{y},\tilde{z})+g(t),\qquad w=\tilde{w}(\tilde{x},\tilde{y},\tilde{z})+h(t),$$

$$p=\tilde{p}(\tilde{x},\tilde{y},\tilde{z})-xf'(t)-yg'(t)-zh'(t)+k(t),$$

where $F'=f$, $G'=g$, $H'=h$, $k=\frac{1}{2}[f^2+g^2+h^2]+\int j\,dt$.

The invariants of the dilatation subgroup (obtained from (3.4) with $\beta\neq 0$ and all other parameters and functions vanishing) are found to be

$$\eta_1=\tilde{y}/\tilde{x},\qquad \eta_2=\tilde{z}/\tilde{x}$$

and

$$\tilde{u}=\tilde{x}^{-1}\Gamma(\eta_1,\eta_2),\qquad \tilde{v}=\tilde{x}^{-1}\Lambda(\eta_1,\eta_2),$$

$$\tilde{w}=\tilde{x}^{-1}\Phi(\eta_1,\eta_2),\qquad \tilde{p}=\tilde{x}^{-2}\Omega(\eta_1,\eta_2).$$

where Γ, Λ, Φ, and Ω satisfy the partial differential equations

$$-\Gamma^2-\eta_1\Gamma\Gamma_{\eta_1}-\eta_2\Gamma\Gamma_{\eta_2}+\Lambda\Gamma_{\eta_1}+\Phi\Gamma_{\eta_2}-2\Omega-\eta_1\Omega_{\eta_1}-\eta_2\Omega_{\eta_2}$$
$$-\mu\left(2\Gamma+4\eta_1\Gamma_{\eta_1}+4\eta_2\Gamma_{\eta_2}+\Gamma_{\eta_1\eta_1}+\Gamma_{\eta_2\eta_2}+\eta_1^2\Gamma_{\eta_1\eta_1}+2\eta_1\eta_2\Gamma_{\eta_1\eta_2}+\eta_2^2\Gamma_{\eta_2\eta_2}\right)=0,$$

$$-\Gamma\Lambda-\eta_1\Gamma\Lambda_{\eta_1}-\eta_2\Gamma\Lambda_{\eta_2}+\Lambda\Lambda_{\eta_1}+\Phi\Lambda_{\eta_2}+\Omega_{\eta_1}-\mu(2\Lambda+4\eta_1\Lambda_{\eta_1}$$
$$+4\eta_2\Lambda_{\eta_2}+\Lambda_{\eta_1\eta_1}+\Lambda_{\eta_2\eta_2}+\eta_1^2\Lambda_{\eta_1\eta_1}+2\eta_1\eta_2\Lambda_{\eta_1\eta_2}+\eta_2^2\Lambda_{\eta_2\eta_2}\big)=0, \qquad (5.1)$$

$$-\Gamma\Phi-\eta_1\Gamma\Phi_{\eta_1}-\eta_2\Gamma\Phi_{\eta_2}+\Lambda\Phi_{\eta_2}+\Phi\Phi_{\eta_2}+\Omega_{\eta_2}-\mu\left(2\Phi+4\eta_1\Phi_{\eta_1}\right.$$
$$\left.+4\eta_2\Phi_{\eta_2}+\Phi_{\eta_1\eta_1}+\Phi_{\eta_2\eta_2}+\eta_1^2\Phi_{\eta_1\eta_1}+2\eta_1\eta_2\Phi_{\eta_1\eta_2}+\eta_2^2\Phi_{\eta_2\eta_2}\right)=0,$$

$$-\Gamma-\eta_1\Gamma_{\eta_1}-\eta_2\Gamma_{\eta_2}+\Lambda_{\eta_1}+\Phi_{\eta_2}=0.$$

However, no further group reduction is possible. But, system (5.1) is reduced to the system of ordinary differential equations

$$-\Gamma^2-\eta\Gamma\Gamma_\eta+\Lambda\Gamma_\eta-\Phi\Gamma_\eta-2\Omega-\eta\Gamma_\eta-\mu\left(2\Gamma+4\eta\Gamma_\eta+2\Gamma_{\eta\eta}+\eta^2\Gamma_{\eta\eta}\right)=0, \quad (5.2a)$$

$$-\Gamma\Lambda-\eta\Gamma\Lambda_\eta+\Lambda\Lambda_\eta-\Phi\Lambda_\eta+\Omega_\eta-\mu\left(2\Lambda+4\eta\Lambda_\eta+2\Lambda_{\eta\eta}+\eta^2\Lambda_{\eta\eta}\right)=0, \quad (5.2b)$$

$$-\Gamma\Phi-\eta\Gamma\Phi_\eta+\Lambda\Phi_\eta-\Phi\Phi_\eta-\Omega_\eta-\mu\left(2\Phi+4\eta\Phi_\eta+2\Phi_{\eta\eta}+\eta^2\Phi_{\eta\eta}\right)=0, \quad (5.2c)$$

$$-\Gamma-\eta\Gamma_\eta+\Lambda_\eta-\Phi_\eta=0. \quad (5.2d)$$

in the variable $\eta=\eta_1-\eta_2$. The last of these is satisfied when

$$\Lambda-\Phi=\eta\Gamma-c_1, \quad (5.3)$$

where c_1 is an arbitrary constant. Two equations free of Λ's and Φ's may now be obtained:

$$-\Gamma^2-c_1\Gamma_\eta-2\Omega-\eta\Omega_\eta-\mu\left(2\Gamma+4\eta\Gamma_\eta+2\Gamma_{\eta\eta}+\eta^2\Gamma_{\eta\eta}\right)=0, \quad (5.4)$$

by substituting (5.3) into (5.2a), and

$$-\eta\Gamma^2+2\Omega_\eta-\mu\left(6\eta\Gamma+4\Gamma_\eta+6\eta^2\Gamma_\eta+2\eta\Gamma_{\eta\eta}+\eta^3\Gamma_{\eta\eta}\right)=0, \quad (5.5)$$

by substituting (5.3) into (5.2b) and (5.2c) and then subtracting. Solving (5.5) for Ω_η and replacing for Ω_η in (5.4) yields

$$\Omega=\tfrac{1}{2}\Big[-\Gamma^2-\tfrac{1}{2}\eta^2\Gamma^2-c_1\Gamma_\eta-\mu\big(2\Gamma+3\eta^2\Gamma+6\eta\Gamma_\eta$$
$$+3\eta^3\Gamma_\eta+2\Gamma_{\eta\eta}+2\eta^2\Gamma_{\eta\eta}+\tfrac{1}{2}\eta^4\Gamma_{\eta\eta}\big)\Big]. \quad (5.6)$$

Equations (5.5) and (5.6) imply that

$$2\eta\Gamma^2+2\Gamma\Gamma_\eta+\eta^2\Gamma\Gamma_\eta+c_1\Gamma_{\eta\eta}+\mu\big(12\eta\Gamma+12\Gamma_\eta+18\eta^2\Gamma_\eta$$
$$12\eta\Gamma_{\eta\eta}+6\eta^3\Gamma_{\eta\eta}+2\Gamma_{\eta\eta\eta}+2\eta^2\Gamma_{\eta\eta\eta}+\tfrac{1}{2}\eta^4\Gamma_{\eta\eta\eta}\big)=0. \quad (5.7)$$

One solution of (5.7) is

$$\Gamma=-6\mu. \quad (5.8)$$

The corresponding values for Λ and $(\Lambda-\Phi)$ from (5.6) and (5.3) are

$$\Lambda=-12\mu^2, \quad (5.9)$$

$$\Lambda-\Phi=-6\mu\eta-c_1. \quad (5.10)$$

Substitution of (5.8)–(5.10) into (5.2c) results in

$$4\Phi - \left[\frac{c_1}{\mu} + 4\eta\right]\Phi_\eta - (\eta^2+2)\Phi_{\eta\eta} = 0,$$

whose general solution, for the case $c_1 = 0$, is

$$\Phi = c_2\mu\eta - c_3\mu\left[\tfrac{1}{4} + \tfrac{1}{8}\eta^2(\eta^2+2)^{-1} - \frac{3\eta}{8\sqrt{2}}\arctan\left[\frac{\eta}{\sqrt{2}}\right]\right].$$

Retracing our steps back to the original variables leads to the time-dependent solution,

$$u = -6\mu(x - F(t))^{-1} + f(t), \tag{5.11a}$$

$$v = \mu\Bigg\{(c_2 - 6)(x - F(t))^{-2}R - c_3\Bigg[(4(x - F(t)))^{-1}$$

$$+ (2(x-F(t)))^{-3}R^2\Bigg((x-F(t))^{-2}(R+2)^{-1} + \frac{3}{8\sqrt{2}}(x-F(t))^{-2}R$$

$$\times\arctan\left(\frac{(x-F(t))^{-1}R}{\sqrt{2}}\right)\Bigg)\Bigg\} + g(t), \tag{5.11b}$$

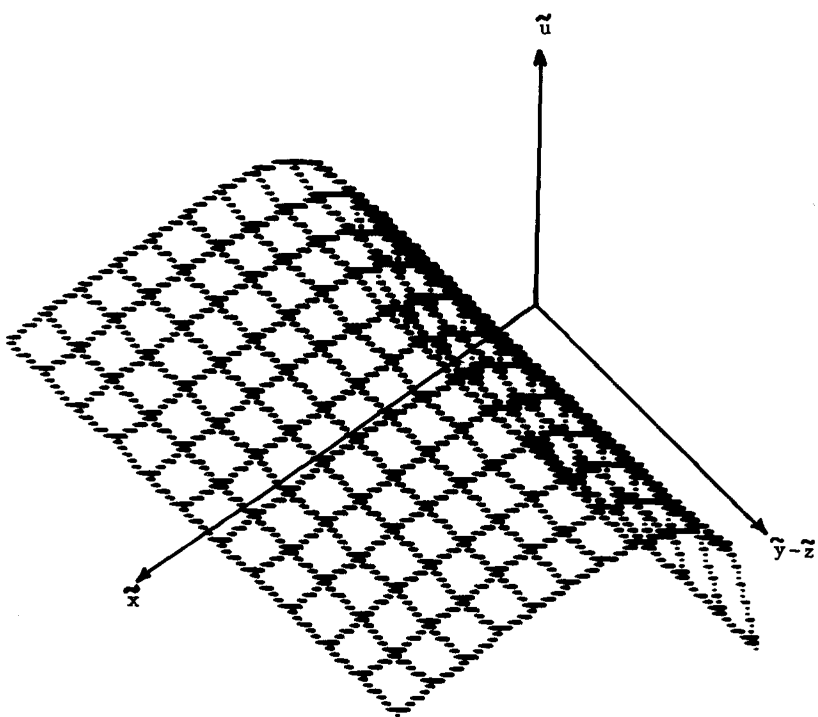

Fig. 1. Graph of equation (5.11a).

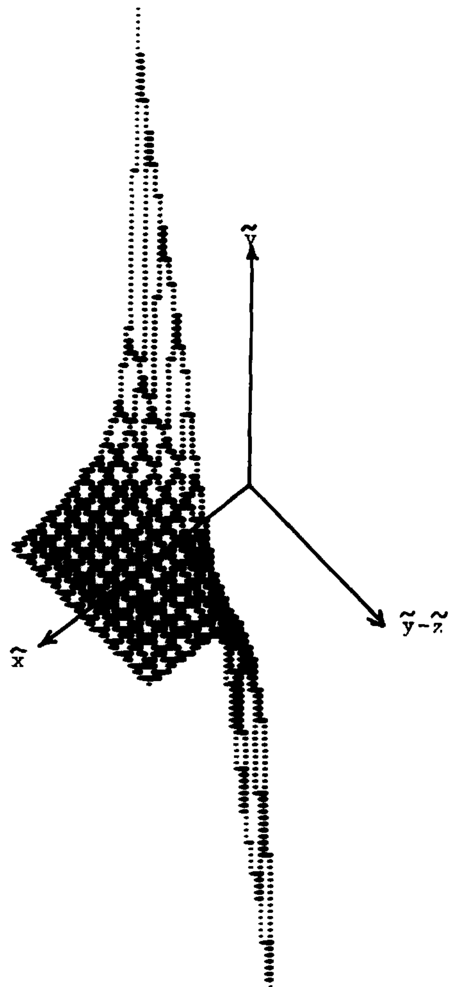

Fig. 2. Graph of equation (5.11b).

$$w=\mu\left\{c_2(x-F(t))^{-2}R-c_3\left[(4(x-F(t)))^{-1}\right.\right.$$

$$+(2(x-F(t)))^{-3}R^2\left((x-F(t))^{-2}(R+2)^{-1}+\frac{3}{8\sqrt{2}}(x-F(t))^{-2}R\right.$$

$$\left.\left.\times\arctan\left[\frac{(x-F(t))^{-1}R}{\sqrt{2}}\right]\right]\right\}+h(t), \qquad (5.11c)$$

$$p=-12\mu^2(x-F(t))^{-2}-xf'(t)-yg'(t)-zh'(t)+k(t), \qquad (5.11d)$$

where $R=y-z-G(t)+H(t)$. These solutions are shown graphically in Figs. 1–4 for the steady-state case (all functions of time identically equal to zero) with $\mu=c_2=c_3=1$.

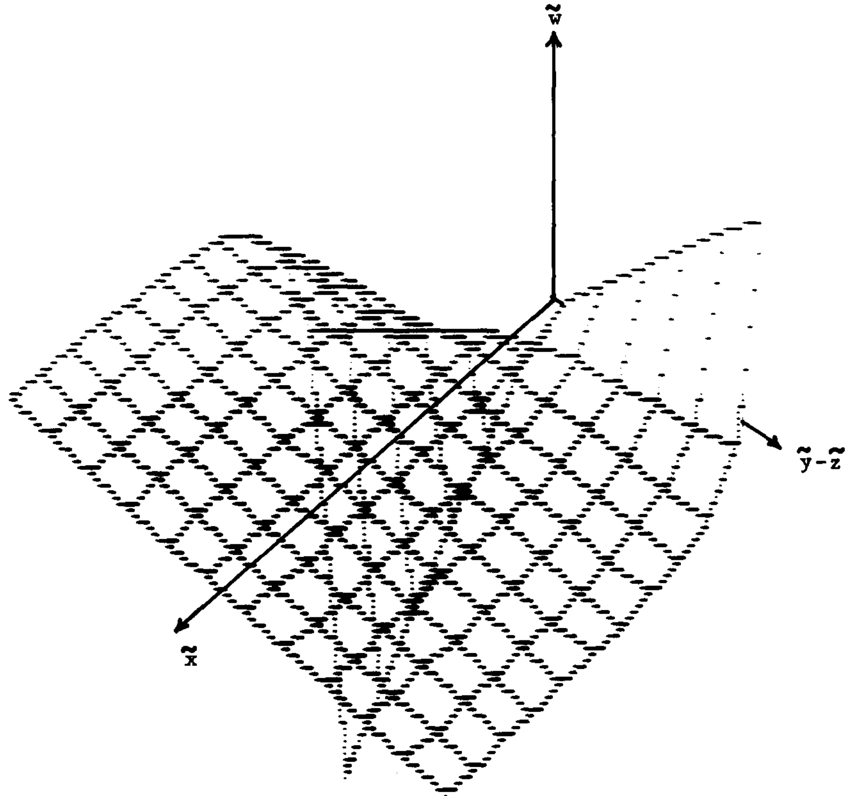

Fig. 3. Graph of equation (5.11c).

A second solution of equation (5.7) is

$$\Gamma = -12\mu\eta^{-2}$$

from which is obtained

$$\Lambda - \Phi = -12\mu\eta^{-1} - c_1, \qquad \Omega = -24\mu^2\eta^{-2}.$$

With these values, equation (5.6c) becomes

$$(\eta^2+2)\Phi_{\eta\eta} + \left[4\eta + \frac{c_1}{\mu}\right]\Phi_\eta + (-12\eta^{-2}+2)\Phi = -48\mu\eta^{-3}. \tag{5.12}$$

When $c_1 = 0$, the general solution of (5.12) is

$$\Phi = \mu\left\{c_2\eta^{-1} + c_3\eta^{-2} + (24+4c_2)\left[-(4\eta^3+8\eta)^{-1} + \frac{3}{4\sqrt{2}}\eta^{-2}\arctan\left[\frac{\eta}{\sqrt{2}}\right]\right]\right\},$$

The solution of (1.1)–(1.4) obtained by going back to the original variables is

$$u = -12\mu(x - F(t))R^{-2} + f(t),$$

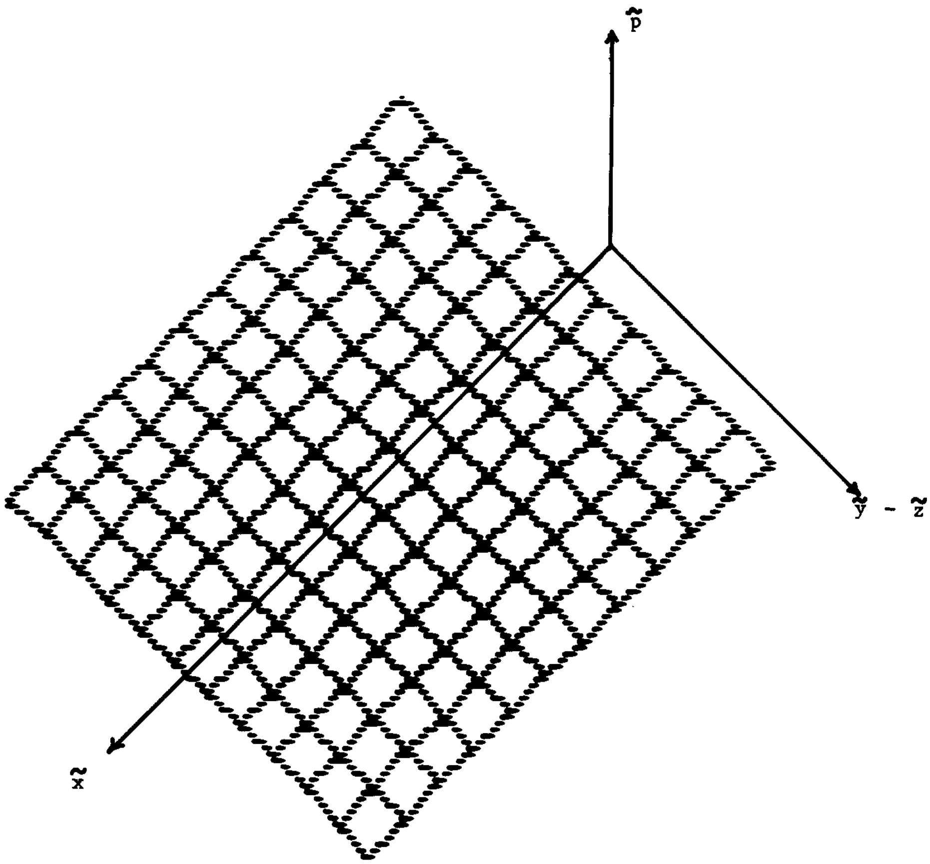

Fig. 4. Graph of equation (5.11d).

$$v=\mu\left\{(c_2-12)R^{-1}+c_3(x-F(t))R^{-2}+(24+4c_2)\left[-\frac{(x-F(t))^2}{4R^3+8(x-F(t))^2R}\right.\right.$$

$$\left.\left.+\frac{3(x-F(t))}{4\sqrt{2}\,R^2}\arctan\left[\frac{R}{\sqrt{2}\,(x-F(t))}\right]\right]\right\}+g(t),$$

$$w=\mu\left\{c_2R^{-1}+c_3(x-F(t))R^{-2}+(24+4c_2)\left[-\frac{(x-F(t))^2}{4R^3+8(x-F(t))^2R}\right.\right.$$

$$\left.\left.+\frac{3(x-F(t))}{4\sqrt{2}\,R^2}\arctan\left[\frac{R}{\sqrt{2}\,(x-F(t))}\right]\right]\right\}+h(t),$$

$$p=-24\mu^2R^{-2}-xf'(t)-yg'(t)-zh'(t)+k(t),$$

where again, $R=y-z-G(t)+H(t)$.

6. Additional time-dependent solutions

An alternative approach originates with a direct application of the dilatation subgroup to the time-dependent equations (1.1)–(1.4). The invariants of the group are

$$\eta_1 = xt^{-1/2}, \qquad \eta_2 = yt^{-1/2}, \qquad \eta_3 = zt^{-1/2}$$

and

$$u = t^{-1/2} f(\eta_1, \eta_2, \eta_3), \qquad v = t^{-1/2} g(\eta_1, \eta_2, \eta_3),$$

$$w = t^{-1/2} h(\eta_1, \eta_2, \eta_3), \qquad p = t^{-1} j(\eta_1, \eta_2, \eta_3),$$

where f, g, h, and j satisfy the equations

$$\begin{aligned}
&-\tfrac{1}{2}\left[f + \eta_1 f_{\eta_1} + \eta_2 f_{\eta_2} + \eta_3 f_{\eta_3}\right] + f f_{\eta_1} + g f_{\eta_2} + h f_{\eta_3} \\
&\quad + j_{\eta_1} - \mu\left(f_{\eta_1\eta_1} + f_{\eta_2\eta_2} + f_{\eta_3\eta_3}\right) = 0, \\
&-\tfrac{1}{2}\left[g + \eta_1 g_{\eta_1} + \eta_2 g_{\eta_2} + \eta_3 g_{\eta_3}\right] + f g_{\eta_1} + g g_{\eta_2} + h g_{\eta_3} \\
&\quad + j_{\eta_2} - \mu\left(g_{\eta_1\eta_1} + g_{\eta_2\eta_2} + g_{\eta_3\eta_3}\right) = 0, \\
&-\tfrac{1}{2}\left[h + \eta_1 h_{\eta_1} + \eta_2 h_{\eta_2} + \eta_3 h_{\eta_3}\right] + f h_{\eta_1} + g h_{\eta_2} + h h_{\eta_3} \\
&\quad + j_{\eta_3} - \mu\left(h_{\eta_1\eta_1} + h_{\eta_2\eta_2} + h_{\eta_3\eta_3}\right) = 0, \\
&f_{\eta_1} + g_{\eta_2} + h_{\eta_3} = 0.
\end{aligned} \tag{6.1}$$

Two separate lengthy analyses of (6.1) lead to two solutions. The first depends upon μ and involves the error function.

$$\begin{aligned}
u &= c_3 t^{-1/2}\,\mathrm{erf}\left[\left(\mu\left(4c_1^2 + 8c_2^2\right)\right)^{-1/2}(2c_2 x - c_1 y - c_1 z)t^{-1/2}\right] \\
&\quad + \left(\frac{1}{2} - \frac{2c_2^2}{c_1^2 + 2c_2^2}\right) x t^{-1} + \left(\frac{c_1 c_2}{c_1^2 + 2c_2^2}\right)(y + z)t^{-1} + c_4 t^{-1/2}, \\
v &= \frac{c_2 c_3}{c_1} t^{-1/2}\,\mathrm{erf}\left[\left(\mu\left(4c_1^2 + 8c_2^2\right)\right)^{-1/2}(2c_2 x - c_1 y - c_1 z)t^{-1/2}\right] \\
&\quad + \left(\frac{c_1 c_2}{c_1^2 + 2c_2^2}\right) x t^{-1} + \left(-\frac{1}{4} + \frac{c_2^2}{c_1^2 + 2c_2^2}\right)(y + z)t^{-1} + \frac{c_2 c_4}{c_1} t^{-1/2},
\end{aligned}$$

$$w = v,$$

$$p = \left\{\left(\frac{1}{8} - \frac{c_2^2}{c_1^2 + 2c_2^2}\right)x^2 + \left(\frac{1}{16} - \frac{c_1^2}{4c_1^2 + 8c_2^2}\right)(y^2 + z^2)\right.$$

$$+\left(\frac{c_1c_2}{c_1^2+2c_2^2}\right)(xy+xz)+\left(\frac{1}{8}-\frac{c_1}{2c_1^2+4c_2^2}\right)yz\Bigg\}t^{-2}+c_5t^{-1}.$$

The second solution,

$$u=\left[\left(\frac{1}{6}-\frac{c_1}{3}\right)x+\left(\frac{5}{6}+\frac{c_1}{3}\right)y+\left(-\frac{1}{3}+\frac{2c_1}{3}\right)z\right]t^{-1}+\tfrac{1}{3}c_2t^{-1/2},$$

$$v=\left[\left(\frac{5}{6}-\frac{c_1}{3}\right)x+\left(\frac{1}{6}+\frac{c_1}{3}\right)y+\left(\frac{1}{3}+\frac{2c_1}{3}\right)z\right]t^{-1}-\tfrac{1}{3}c_2t^{-1/2},$$

$$w=\tfrac{1}{3}(-x+y-z)t^{-1}+\tfrac{1}{3}c_2t^{-1/2},$$

$$p=-\tfrac{1}{3}(x^2+y^2+z^2)t^{-2}+\tfrac{2}{3}(xy-xz+yz)t^{-2}+\tfrac{1}{2}c_2(x-y+z)t^{-3/2}+c_3t^{-1},$$

is independent of the viscosity of the fluid. Details of these calculations may be found in Boisvert [8].

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